Experiments with tableaux in Moca

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Part I

Once upon a time ...

What's a tableau ?

- Automated refutation method:

  Proving P (or $T\ P$) becomes refuting $\neg P$ (or $F\ P$)

- Algorithmically:
  - Put the formula into DNF.
  - Find a contradiction on every set of this DNF.

Expansion rules (with metavariables)

$$\frac{\alpha(A, B)}{\begin{array}{c} A \\ B \end{array}}$$

Formula type

$\alpha(A, B)$:

$$\begin{array}{c} A \wedge B \\ \neg\,(A \vee B) \\ \neg\,(A \Rightarrow B) \end{array}$$

Corresponding sequent rule

$$\frac{\Gamma, A, B \vdash}{\Gamma, A \wedge B \vdash}\ \wedge$$

Expansion rules (with metavariables)

$$\frac{\alpha(A,B)}{\begin{array}{c}A\\B\end{array}} \qquad \frac{\beta(A,B)}{A \qquad B}$$

Formula type

$\beta(A,B)$:

$$\begin{array}{c}A \vee B\\ \neg\,(A \wedge B)\\ A \Rightarrow B\end{array}$$

Corresponding sequent rule

$$\frac{\Gamma, A \vdash \qquad \Gamma, B \vdash}{\Gamma, A \vee B \vdash}\ \vee$$

Expansion rules (with metavariables)

$$\frac{\alpha(A,B)}{\begin{array}{c}A\\B\end{array}} \qquad \frac{\beta(A,B)}{A \qquad B}$$

$$\frac{\gamma(x,A)}{A[x := X]}$$

Formula type

$\gamma(x,A)$:

$$\begin{array}{c}\forall x A\\ \neg\,(\exists x A)\end{array}$$

Corresponding sequent rule

$$\frac{\Gamma, A[x := t] \vdash}{\Gamma, \forall x\ A \vdash}\ \forall$$

Expansion rules (with metavariables)

$$\frac{\alpha(A,B)}{\begin{array}{c}A\\B\end{array}} \qquad \frac{\beta(A,B)}{A \qquad B}$$

$$\frac{\gamma(x,A)}{A[x := X]} \qquad \frac{\delta(x,A)}{A[x := f_{\text{sko}}(\text{args})]}$$

Formula type

$\delta(x,A)$:

$$\begin{array}{c}\exists x A\\ \neg\,(\forall x A)\end{array}$$

Corresponding sequent rule

$$\frac{\Gamma, A[x := c] \vdash}{\Gamma, \exists x\ A \vdash}\ \exists$$

Expansion rules (with metavariables)

$$\frac{\alpha(A,B)}{\begin{array}{c}A\\B\end{array}} \qquad \frac{\beta(A,B)}{A \qquad B}$$

$$\frac{\gamma(x,A)}{A[x := X]} \qquad \frac{\delta(x,A)}{A[x := f_{\text{sko}}(\text{args})]}$$

$$\frac{\begin{array}{c}A\\\neg A\end{array}}{\odot}$$

Formula type

Corresponding sequent rule

$$\frac{}{\Gamma, A, \neg A \vdash}\ \text{ax}$$

An example: Peirce's law $\mathcal{P}$

I want to prove $\mathcal{P} \hat{=} ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$. Therefore I will refute $\neg\mathcal{P}$.

$$\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$$

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$$\frac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{\dfrac{(A \Rightarrow B) \Rightarrow A}{\neg A}}$$

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$$\dfrac{\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A} \quad \neg A}{\neg(A \Rightarrow B) \qquad\qquad A}$$

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$$\dfrac{\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A} \qquad \dfrac{\dfrac{\dfrac{\neg A}{\neg(A \Rightarrow B)}}{A} \qquad A}{\dfrac{\dfrac{A}{\neg B}}{\odot}}}$$

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I want to prove $\mathcal{P} \hat{=} ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$. Therefore I will refute $\neg\mathcal{P}$.

$$\dfrac{\dfrac{\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A}}{\neg A}}{\dfrac{\dfrac{\neg(A \Rightarrow B)}{A}}{\neg B}} \qquad \dfrac{A}{\odot}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, \neg B, \neg A \vdash}}{\neg(A \Rightarrow B), \neg A \vdash} \qquad \dfrac{}{A, \neg A \vdash}}{(A \Rightarrow B) \Rightarrow A, \neg A \vdash}}{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A) \vdash}$$

Some choices

- Branches of tableaux will be sets (Ass Comm Idem)
- Tableaux will be sets of branches (Ass Comm Idem)
- I want to be looking for only one other formula as the opposite of any given formula (involutive)

Propositional tableaux : DNF transformation

```
type boolean = private
  | Btrue | Bfalse

  | Batom of string

  | Bnot of boolean
     begin
          involutive
          distributive (Band, Bor)
          distributive (Bor, Band)
          rule Bnot Btrue -> Bfalse
          rule Bnot Bfalse > Btrue
     end
```

```
| Band of boolean * boolean
      begin
          associative
          idempotent
          commutative
          distributive (Bor)
      end
```

```
| Bor of boolean * boolean
     begin
       associative
       commutative
       idempotent
     end
```

Pros and cons

- 15 lines for a *readable* decision procedure.
- Maximal sharing is not used ...
- Will have to use `-urorder`

Part II

Blackboard singing in the dead of night ...

Part III

Testing

Hilbert's axioms

$$A \Rightarrow (B \Rightarrow A)$$
$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$
$$A \Rightarrow (A \vee B)$$
$$B \Rightarrow (A \vee B)$$
$$(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$$
$$(A \wedge B) \Rightarrow A$$
$$(A \wedge B) \Rightarrow B$$
$$A \Rightarrow (B \Rightarrow (A \wedge B))$$

Other tautologies

$(A \Rightarrow B) \Rightarrow A$	Peirce's law
$A \vee \neg A$	Excluded middle
$(B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$	B axiom
$(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$	C axiom
$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	W axiom
$(\neg\neg A \Rightarrow \neg\neg B) \Rightarrow \neg\neg(A \Rightarrow B)$	TS 2.1.8D

Equivalences

General problem: $x_1 \Leftrightarrow x_2 \ldots \Leftrightarrow x_n \Leftrightarrow x_1 \Leftrightarrow x_2 \ldots \Leftrightarrow x_n$

n	bbns	more	all	bbs	bbs+all
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	8(99)	3(81)	0	17	0
5	–	138(243)	0.5	–	1
6	–	–	12	–	27
7	–	–	305	–	832
8	–	–	7868	–	–

Part IV

First-order tableaux: a work in progress

Here come the quantifiers

- The program should still be a decision procedure ...
- And deal with quantifiers

First-order tableaux: iterative deepening

```
| Ball of term * formula
 begin
 let substitute v u f = ...
 let rec make_new_formulas n f v = ...

  rule Ball (Var x as v, f) ->
   let fs =
     make_new_formulas maxdepth f v in
   List.fold_left (fun x y -> Band (x, y))
    (List.hd fs)
     (List.tl fs)
 end
```

First-order tableaux: naive skolemization

```
| Bex of term * formula
  begin
   rule Bex (Var x as v, f) ->
    substitute v (new_const ()) f
  end
;;
```

Pros and cons

- Moca tableaux are expanded by rightmost innermost rewriting
- But Smullyan's tableaux are expanded by leftmost outermost rewriting ...
- Backtracking is not easy with normalization
- Inner and outer skolemization are not possible with inner rewrite steps
- Hilbert's ϵ terms can be used! (thanks to delayed substitution). (Not yet implemented)

And now?

- Use varyadic tableaux to find bugs in moca !
- Finish first-order tableaux (find bigger examples)
- Deduction modulo?